SET THEORY HOMEWORK 6

Due Wednesday, December 18.

Problem 1. *Suppose that $\mathbb{P}$ is a poset, $A \subset \mathbb{P}$ is a maximal antichain, $\phi(x)$ is a formula, and $\langle \tau_p \mid p \in A \rangle$ are $\mathbb{P}$ names, such that for all $p \in A$, $p \Vdash \phi(\tau_p)$. Show that there is a $\mathbb{P}$ name τ, such that $1_{\mathbb{P}} \Vdash \phi(\tau)$.*

Problem 2. *Let T be a normal Suslin tree, and let $(\mathbb{P}_T, <) = (T, >)$. Show that although $\mathbb{P}_T$ has the countable chain condition, $\mathbb{P}_T \times \mathbb{P}_T$ does not. Hint: for every $x \in T$, pick two immediate successors p_x, q_x of x. Look at the set $\{(p_x, q_x) \mid x \in T\} \subset \mathbb{P}_T \times \mathbb{P}_T$.*

Problem 3. *Suppose that $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-chain condition. Show that $\mathbb{P}$ has the κ-chain condition, and $1_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ has the κ-chain condition".*

Remark 1. The converse is also true.

Problem 4. *Let $\mathbb{P}$ be a poset such that for every $p \in \mathbb{P}$, there are incompatible $q, r \leq p$. Suppose G is $\mathbb{P}$-generic. Show that $G \times G$ is not $\mathbb{P} \times \mathbb{P}$-generic.*

Problem 5. *Let $S \subset \omega_1$ be a stationary set. Define $\mathbb{P} := \{p \subset S \mid p \text{ is closed and bounded}\}$, and set $p \leq q$ if p end extends q i.e. for some α, $p \cap \alpha = q$.*

1. *Show that $\mathbb{P}$ is ω-distributive, i.e. if $p \Vdash \dot{f} : \omega \to ON$, then there is some $q \leq p$ and a function g in the ground model, such that $q \Vdash \dot{f} = \check{g}$. Note that this implies that $\mathbb{P}$ adds no countable subsets of ω_1, and hence it preserves ω_1.*
2. *What is the best chain condition for $\mathbb{P}$? Justify your answer. Use that and the above to show that $\mathbb{P}$ preserves all cardinals.*
3. *Suppose that $T := S \setminus \omega_1$ is also stationary. Let G be a $\mathbb{P}$-generic filter. Show that in $V[G]$, T is nonstationary.*